This Appendix presents a matrix algebra representation of the geodesic equation of the General Theory of Relativity. For the spacetime vector $\mathbf{x} = \left(x^1, x^2, x^3, x^4\right)^T$, the classic, "double index summation" representation of the geodesic equation is given as

$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} = 0 \tag{1}$$

Due to the double index summation rules, the second term is actually a quadratic form, and can be written using matrix algebra. We can then write Eq. (1) as

$$\frac{d^2 x^i}{d\tau^2} = -\frac{d\mathbf{x}^T}{d\tau} \mathbf{\Gamma}^{x^i} \frac{d\mathbf{x}}{d\tau} \tag{2}$$

where the *Christoffel matrix* $\mathbf{\Gamma}^{x^i}$ is given as

$$\mathbf{\Gamma}^{x_i} = \begin{bmatrix} \Gamma^{x_i}_{x_1 x_1} & \Gamma^{x_i}_{x_1 x_2} & \Gamma^{x_i}_{x_1 x_3} & \Gamma^{x_i}_{x_1 x_4} \\ \Gamma^{x_i}_{x_2 x_1} & \Gamma^{x_i}_{x_2 x_2} & \Gamma^{x_i}_{x_2 x_3} & \Gamma^{x_i}_{x_2 x_4} \\ \Gamma^{x_i}_{x_3 x_1} & \Gamma^{x_i}_{x_3 x_2} & \Gamma^{x_i}_{x_3 x_3} & \Gamma^{x_i}_{x_3 x_4} \\ \Gamma^{x_i}_{x_4 x_1} & \Gamma^{x_i}_{x_4 x_2} & \Gamma^{x_i}_{x_4 x_3} & \Gamma^{x_i}_{x_4 x_4} \end{bmatrix} \tag{3}$$

(Here, we have adopted a style of explicitly writing out the raised and lowered indices as the elements of **x** they represent. As we shall see, this will make identification easier when we adopt some specific coordinate system, such as the spherical polar coordinate system.) Each element of $\mathbf{\Gamma}^{x_i}$ is a *Christoffel symbol of the second kind*, which is a scalar and can be computed as a dot product as follows:

$$\Gamma^{x_i}_{x_j x_k} = \frac{1}{2} \mathbf{g}^{x_i} \bullet \left(\frac{\partial \mathbf{g}_{x_k}}{\partial x_j} + \frac{\partial \mathbf{g}_{x_j}}{\partial x_k} - \frac{\partial g_{x_j x_k}}{\partial \mathbf{x}} \right) \tag{4}$$

(For most gravitational fields that make “physical sense,” the Christoffel matrix is symmetric, i.e., $\Gamma^{x_i}_{x_k x_j} = \Gamma^{x_i}_{x_j x_k}$.) The **g** vectors in the above are extracted out of the 4 x 4 *metric matrix* **G** and its inverse, as will be exemplified in a moment.

The elements of **G** are functions of the elements of **x** and give the exact metrical structure of the geometry that represents the gravitational field. The example for a Kerr field is instructive. We set the spacetime vector equal to the spherical polar coordinates and coordinate time *t* of a Cartesian laboratory coordinate frame centered on a spherically shaped central mass, i.e., we set $\mathbf{x} = (x_1, x_2, x_3 x_4)^T = (r, \theta, \phi, t)^T$. Then, for a Kerr field, the metric tensor (matrix) is

$$\mathbf{G} = \begin{bmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & g_{\theta\theta} & 0 & 0 \\ 0 & 0 & g_{\phi\phi} & g_{\phi t} \\ 0 & 0 & g_{t\phi} & g_{tt} \end{bmatrix} \qquad (6)$$

where

$$g_{rr} = -\frac{\Sigma}{c^2 \Delta}, \quad g_{\theta\theta} = -\frac{\Sigma}{c^2}, \quad g_{\phi\phi} = -\frac{1}{c^2}\left[\frac{r^2 + a^2 \ {}^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta,$$

$$g_{tt} = \frac{\Delta - a^2 \sin^2\theta}{\Sigma}, \quad g_{t\phi} = g_{\phi t} = \frac{1}{c^2}\left[\frac{a \sin^2\theta \ \ r^2 + a^2 - \Delta}{\Sigma}\right]$$

with

$$\Sigma = r^2 + a^2 \cos^2\theta, \quad \text{and} \quad \Delta = r^2 + a^2 - r_S r$$

As usual, *c* is the speed of light (in a vacuum) and *G* is Newton’s gravitational constant. The “characteristic length” r_S is the Schwarzschild radius, where $r_S = 2GM/c^2$ and *M* is the rest mass of the central body. The characteristic length associated with the angular momentum of the central body (assuming it is spinning with its “roll axis” coincident with the *z*-axis of the laboratory frame) is given by $a = Jc^2/GM$, where *J* is the central body’s angular momentum. (If the central mass is not spinning, so that $J = 0$ and therefore $a = 0$, the elements in Eq. (6) describe a Schwarzschild field. Specifically, the off diagonal elements $g_{\phi t}$ and $g_{t\phi}$ are then zero, and the Schwarzschild metric matrix is diagonal when written in polar coordinates.)

Equation (4) is used is as follows. Let's set $i = 1$, (i.e., concentrate on $x_1 = r$), and derive $d^2 r / d\tau^2$. We need to derive each Christoffel symbol in Eq. (3). Using Eq. (4), and setting its $x_j = x_k = x_1 = r$, we see that the (1,1) element of $\mathbf{\Gamma}^{x_1} = \mathbf{\Gamma}^r$ is

$$\begin{aligned}\Gamma^r_{rr} &= \frac{1}{2}\mathbf{g}^r \bullet \left(\frac{\partial \mathbf{g}_r}{\partial r} + \frac{\partial \mathbf{g}_r}{\partial r} - \frac{\partial g_{rr}}{\partial \mathbf{x}} \right) \\ &= \frac{1}{2}\mathbf{g}^r \bullet \left(2\frac{\partial \mathbf{g}_r}{\partial r} - \frac{\partial g_{rr}}{\partial \mathbf{x}} \right)\end{aligned} \tag{7}$$

In a computer simulation, all of these equations are evaluated "at a point in the space of the laboratory, at a point in laboratory time." At some given point in coordinate (laboratory) time t, the test particle coasting along the geodesic will have some particular $\left(r, \theta, \phi, \ dr/dt, d\theta/dt, d\phi/dt\right)$ and the elements of **G** can be evaluated using Eq. (6). Once **G** is known, it can be numerically inverted. Given the ordering of the coordinates adopted here (i.e., r is the first coordinate, θ is the second, etc.), in Eq. (7), the vector $\mathbf{g}^r$ equals the extracted first row of this inverse matrix. Because of the block diagonal structure of **G** for a Kerr field, the inverse of **G** also possesses the same block diagonal structure, and the first row of the inverse is $\mathbf{g}^r = \left(g^{rr}, 0, 0, 0\right)$ where we have used g^{rr} to denote the (1,1) element of the inverse of **G**. (Note also that due to the block diagonal structure of **G**, g^{rr} simply equals $1/g_{rr}$, with g_{rr} given in Eq. (6).) In Eq. (7), the vector $\mathbf{g}_r$ refers to the first row of the non-inverted **G**. We see, therefore, that the Kerr Christoffel symbol Γ^r_{rr} can be written as

$$\Gamma^r_{rr} = \frac{1}{2}\begin{pmatrix} g^{rr} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2\frac{\partial g_{rr}}{\partial r} & - & \frac{\partial g_{rr}}{\partial r} \\ 0 & - & \frac{\partial g_{rr}}{\partial \theta} \\ 0 & - & 0 \\ 0 & - & 0 \end{pmatrix} = \frac{1}{2} g^{rr}\frac{\partial g_{rr}}{\partial r} \tag{8}$$

In the above, we have written the sum of the three vectors $\left(\frac{\partial \mathbf{g}_r}{\partial r} + \frac{\partial \mathbf{g}_r}{\partial r} - \frac{\partial g_{rr}}{\partial \mathbf{x}} = 2\frac{\partial \mathbf{g}_r}{\partial r} - \frac{\partial g_{rr}}{\partial \mathbf{x}}\right)$ as a column vector to take advantage of the way a dot product can be written in the form $\mathbf{x} \bullet \mathbf{y} = \mathbf{x}^T \mathbf{y}$. We believe this makes the exact matrix algebra operations to be performed more clear. (Note that none of the Kerr metric matrix elements are functions of ϕ and/or t, so all derivatives with respect to these coordinates (e.g., $\partial g_{rr}/\partial\phi$ and $\partial g_{rr}/\partial t$) are zero.)

In an analogous manner, the Kerr Christoffel symbol $\Gamma^r_{r\theta}$ can be derived as

$$\Gamma^r_{r\theta} = \frac{1}{2}\mathbf{g}^r \bullet \left(\frac{\partial \mathbf{g}_\theta}{\partial r} + \frac{\partial \mathbf{g}_r}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial \mathbf{x}}\right)$$

$$= \frac{1}{2}\begin{pmatrix} g^{rr} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & + & \frac{\partial g_{rr}}{\partial \theta} & - & \frac{\partial g_{r\theta}}{\partial r} \\ \frac{\partial g_{\theta\theta}}{\partial r} & + & 0 & - & \frac{\partial g_{r\theta}}{\partial \theta} \\ 0 & + & 0 & - & 0 \\ 0 & + & 0 & - & 0 \end{pmatrix} = \frac{1}{2} g^{rr}\frac{\partial g_{rr}}{\partial \theta} \tag{9}$$

(The off-diagonal element $g_{r\theta}$ is equal to zero for a Kerr field, hence in the above, $\partial g_{r\theta}/\partial r = \partial g_{r\theta}/\partial\theta = 0$.) Further matrix algebra yields the entire symmetric Christoffel matrix for computing the Kerr relativistic radial acceleration as

$$
\mathbf{\Gamma}^r = \begin{bmatrix}
\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} & \frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial \theta} & 0 & 0 \\
\frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial \theta} & -\frac{1}{2} g^{rr} \frac{\partial g_{\theta\theta}}{\partial r} & 0 & 0 \\
0 & 0 & -\frac{1}{2} g^{rr} \frac{\partial g_{\phi\phi}}{\partial r} & -\frac{1}{2} g^{rr} \frac{\partial g_{\phi t}}{\partial r} \\
0 & 0 & -\frac{1}{2} g^{rr} \frac{\partial g_{t\phi}}{\partial r} & -\frac{1}{2} g^{rr} \frac{\partial g_{tt}}{\partial r}
\end{bmatrix} \tag{10}
$$

We can now derive $d^2r/d\tau^2$ as

$$
\begin{aligned}
\frac{d^2 r}{d\tau^2} &= -\frac{d\mathbf{x}^T}{d\tau} \mathbf{\Gamma}^r \frac{d\mathbf{x}}{d\tau} \\
&= -\begin{pmatrix} \frac{dr}{d\tau} & \frac{d\theta}{d\tau} & \frac{d\phi}{d\tau} & \frac{dt}{d\tau} \end{pmatrix} \mathbf{\Gamma}^r \begin{pmatrix} \frac{dr}{d\tau} \\ \frac{d\theta}{d\tau} \\ \frac{d\phi}{d\tau} \\ \frac{dt}{d\tau} \end{pmatrix} \\
&= -\frac{1}{2} g^{rr} \left[\frac{\partial g_{rr}}{\partial r} \left(\frac{dr}{d\tau} \right)^2 - \frac{\partial g_{\theta\theta}}{\partial r} \left(\frac{d\theta}{d\tau} \right)^2 - \frac{\partial g_{\phi\phi}}{\partial r} \left(\frac{d\phi}{d\tau} \right)^2 - \frac{\partial g_{tt}}{\partial r} \left(\frac{dt}{d\tau} \right)^2 \right. \\
&\qquad \left. + 2 \frac{\partial g_{rr}}{\partial \theta} \frac{dr}{d\tau} \frac{d\theta}{d\tau} - 2 \frac{\partial g_{\phi t}}{\partial r} \frac{d\phi}{d\tau} \frac{dt}{d\tau} \right]
\end{aligned} \tag{11}
$$

The last line in Eq. (11) is an expansion of the matrix algebra, but when numerically implementing these equations on a computer it is best to set up a 4×4 matrix and populate it with the appropriate individually computed Christoffel symbols (e.g., like in Eq. (10)), and then compute the coordinate acceleration as a quadratic form with matrix algebra subroutines.

Kerr-specific relativistic equations for $d^2\theta/d\tau^2$, $d^2\phi/d\tau^2$ and $d^2t/d\tau^2$ can also be derived. For θ we find

$$\frac{d^2\theta}{d\tau^2} = -\frac{d\mathbf{x}^T}{d\tau}\mathbf{\Gamma}^\theta \frac{d\mathbf{x}}{d\tau} \tag{12}$$

where

$$\mathbf{\Gamma}^\theta = \begin{bmatrix} -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{rr}}{\partial\theta} & \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} & 0 & 0 \\ \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} & \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\phi\phi}}{\partial\theta} & -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{\phi t}}{\partial\theta} \\ 0 & 0 & -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{t\phi}}{\partial\theta} & -\frac{1}{2}g^{\theta\theta}\frac{\partial g_{tt}}{\partial\theta} \end{bmatrix}$$

For ϕ we find

$$\frac{d^2\phi}{d\tau^2} = -\frac{d\mathbf{x}^T}{d\tau}\mathbf{\Gamma}^\phi \frac{d\mathbf{x}}{d\tau} \tag{13}$$

where

$$\mathbf{\Gamma}^\phi = \begin{bmatrix} 0 & 0 & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial r} + \frac{1}{2}g^{\phi t}\frac{\partial g_{\phi t}}{\partial r} & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{t\phi}}{\partial r} + \frac{1}{2}g^{\phi t}\frac{\partial g_{tt}}{\partial r} \\ 0 & 0 & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial\theta} + \frac{1}{2}g^{\phi t}\frac{\partial g_{\phi t}}{\partial\theta} & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{t\phi}}{\partial\theta} + \frac{1}{2}g^{\phi t}\frac{\partial g_{tt}}{\partial\theta} \\ \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial r} + \frac{1}{2}g^{\phi t}\frac{\partial g_{\phi t}}{\partial r} & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial\theta} + \frac{1}{2}g^{\phi t}\frac{\partial g_{\phi t}}{\partial\theta} & 0 & 0 \\ \frac{1}{2}g^{\phi\phi}\frac{\partial g_{t\phi}}{\partial r} + \frac{1}{2}g^{\phi t}\frac{\partial g_{tt}}{\partial r} & \frac{1}{2}g^{\phi\phi}\frac{\partial g_{t\phi}}{\partial\theta} + \frac{1}{2}g^{\phi t}\frac{\partial g_{tt}}{\partial\theta} & 0 & 0 \end{bmatrix}$$

For t we find

$$\frac{d^2 t}{d\tau^2} = -\frac{d\mathbf{x}^T}{d\tau}\mathbf{\Gamma}^t \frac{d\mathbf{x}}{d\tau} \tag{14}$$

where

$$\Gamma^t = \begin{bmatrix} 0 & 0 & \frac{1}{2} g^{t\phi} \frac{\partial g_{\phi\phi}}{\partial r} + \frac{1}{2} g^{tt} \frac{\partial g_{\phi t}}{\partial r} & \frac{1}{2} g^{t\phi} \frac{\partial g_{t\phi}}{\partial r} + \frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial r} \\ 0 & 0 & \frac{1}{2} g^{t\phi} \frac{\partial g_{\phi\phi}}{\partial \theta} + \frac{1}{2} g^{tt} \frac{\partial g_{\phi t}}{\partial \theta} & \frac{1}{2} g^{t\phi} \frac{\partial g_{t\phi}}{\partial \theta} + \frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial \theta} \\ \frac{1}{2} g^{t\phi} \frac{\partial g_{\phi\phi}}{\partial r} + \frac{1}{2} g^{tt} \frac{\partial g_{\phi t}}{\partial r} & \frac{1}{2} g^{t\phi} \frac{\partial g_{\phi\phi}}{\partial \theta} + \frac{1}{2} g^{tt} \frac{\partial g_{\phi t}}{\partial \theta} & 0 & 0 \\ \frac{1}{2} g^{t\phi} \frac{\partial g_{t\phi}}{\partial r} + \frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial r} & \frac{1}{2} g^{t\phi} \frac{\partial g_{t\phi}}{\partial \theta} + \frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial \theta} & 0 & 0 \end{bmatrix}$$

Equations (11), (12), (13) and (14) constitute the Kerr-specific spherical polar coordinate relativistic accelerations. These can be programmed and numerically integrated (using, e.g., a 4^{th} order Runge-Kutta routine) in order to generate the full three dimensional relativistic Kerr geodesic motion of a test particle in a non-rotating Cartesian laboratory coordinate frame centered on a central body. (The partial derivatives in each of these equations can be derived analytically by hand or by using a symbolic manipulation program capable of analytically deriving derivatives, or they can be evaluated numerically.)