This Appendix presents the analytical equations for the partial derivatives of the elements of the Kerr metric matrix (tensor) written in polar coordinates (r, θ, ϕ, t), and matrix equations for computing the inverse of the Kerr metric matrix. Using the proper time as the affine parameter, the Kerr metric matrix is given as

$$\mathbf{G} = \begin{bmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & g_{\theta\theta} & 0 & 0 \\ 0 & 0 & g_{\phi\phi} & g_{\phi t} \\ 0 & 0 & g_{t\phi} & g_{tt} \end{bmatrix} \tag{1}$$

where

$$\begin{aligned} g_{rr} &= -\frac{\Sigma}{c^2\Delta}, \quad g_{\theta\theta} = -\frac{\Sigma}{c^2} \quad g_{\phi\phi} = -\frac{1}{c^2}\left[\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right]\sin^2\theta \\ g_{tt} &= \frac{\Delta - a^2\sin^2\theta}{\Sigma}, \quad g_{\phi t} = g_{t\phi} = \frac{1}{c^2}\left[\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma}\right] \end{aligned} \tag{2}$$

with

$$\Sigma = r^2 + a^2\cos^2\theta \quad \text{and} \quad \Delta = r^2 - 2mr + a^2$$

and $m = GM/c^2$. As usual, c is the speed of light (in a vacuum) and G is Newton's gravitational constant. M is the mass of the central body. The characteristic length associated with the angular momentum of the central body (assuming it is spinning with its "roll axis" coincident with the z-axis of the laboratory frame) is given by $a = J/(Mc)$, where J is the central body's angular momentum. (If the central mass is not spinning, so that $J = 0$ and therefore $a = 0$, the elements in Eq. (2) describe a Schwarzschild field. Specifically, the off diagonal elements $g_{\phi t}$ and $g_{t\phi}$ are then zero, and the Schwarzschild metric matrix is diagonal when written in polar coordinates.) We also see from Eq. (2) that no elements are functions of ϕ and/or t, so that all partial derivatives with respect to these coordinates are zero.

We begin by first deriving the partials of Σ and Δ:

$$\frac{\partial \Sigma}{\partial r} = 2r, \qquad \frac{\partial \Sigma}{\partial \theta} = -2a^2 \cos\theta \sin\theta,$$
$$\frac{\partial \Delta}{\partial r} = 2(r-m), \qquad \frac{\partial \Delta}{\partial \theta} = 0 \tag{3}$$

We can now proceed to compute each required derivative. The final result for each derivative presented here may not be as fully reduced as possible, but the goal of these derivations is to produce equations that are to be programmed on a computer, and the results presented here are more than sufficient for this purpose. Beginning with the first element of the Kerr metric matrix, we see

$$\begin{aligned}
\frac{\partial g_{rr}}{\partial r} &= \frac{\partial}{\partial r}\left(-\frac{\Sigma}{c^2\Delta}\right) \\
&= -\frac{1}{c^2}\frac{\partial}{\partial r}\left(\Delta^{-1}\Sigma\right) \\
&= -\frac{1}{c^2}\left(\Delta^{-1}\frac{\partial \Sigma}{\partial r} - \Sigma\Delta^{-2}\frac{\partial \Delta}{\partial r}\right) \\
&= \frac{1}{c^2\Delta}\left(\frac{\Sigma}{\Delta}\frac{\partial \Delta}{\partial r} - \frac{\partial \Sigma}{\partial r}\right)
\end{aligned}$$

For an actual computation in an orbit simulation, the values of Σ, Δ and the partials in Eq. (3) are first computed, and then used to compute a derivative value such as that given above.

We continue with the partial differentiation:

$$\begin{aligned}
\frac{\partial g_{rr}}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-\frac{\Sigma}{c^2\Delta}\right) \\
&= -\frac{1}{c^2}\frac{\partial}{\partial \theta}\left(\Delta^{-1}\Sigma\right) \\
&= -\frac{1}{c^2}\left(\Delta^{-1}\frac{\partial \Sigma}{\partial \theta} - \Sigma\Delta^{-2}\frac{\partial \Delta}{\partial \theta}\right) \\
&= -\frac{1}{c^2\Delta}\frac{\partial \Sigma}{\partial \theta}
\end{aligned}$$

$$\frac{\partial g_{\theta\theta}}{\partial r} = \frac{\partial}{\partial r}\left(-\frac{\Sigma}{c^2}\right)$$
$$= -\frac{1}{c^2}\frac{\partial \Sigma}{\partial r}$$

$$\frac{\partial g_{\theta\theta}}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-\frac{\Sigma}{c^2}\right)$$
$$= -\frac{1}{c^2}\frac{\partial \Sigma}{\partial \theta}$$

$$\frac{\partial g_{\phi\phi}}{\partial r} = \frac{\partial}{\partial r}\left[-\frac{1}{c^2}\left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta\right]$$
$$= -\frac{1}{c^2}\frac{\partial}{\partial r}\left[\Sigma^{-1}\left(\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta\right)\sin^2\theta\right]$$
$$= -\frac{\sin^2\theta}{c^2}\frac{\partial}{\partial r}\left[\Sigma^{-1}\left(\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta\right)\right]$$
$$= \frac{\sin^2\theta}{c^2}\left[\left(\frac{\left(r^2+a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma^2}\right)\frac{\partial \Sigma}{\partial r}\right.$$
$$\left. - \frac{4r\left(r^2+a^2\right)}{\Sigma} + \frac{a^2\sin^2\theta}{\Sigma}\frac{\partial \Delta}{\partial r}\right]$$

$$\frac{\partial g_{\phi\phi}}{\partial\theta} = \frac{\partial}{\partial\theta}\left[-\frac{1}{c^2}\left(\frac{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta\right]$$

$$= -\frac{1}{c^2}\frac{\partial}{\partial\theta}\left[\Sigma^{-1}\left(\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta\right)\sin^2\theta\right]$$

$$= -\frac{1}{c^2}\left[\Sigma^{-1}\left(\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta\right)(2)\sin\theta\cos\theta\right.$$

$$+\sin^2\theta\left(\Sigma^{-1}(-1)\Delta a^2(2)\sin\theta\cos\theta\right.$$

$$\left.\left.+\left(\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta\right)(-1)\Sigma^{-2}\frac{\partial\Sigma}{\partial\theta}\right)\right]$$

$$= \frac{\sin\theta}{c^2\Sigma}\left[\sin\theta\frac{\partial\Sigma}{\partial\theta}\left(\frac{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}{\Sigma}-\Delta\right)\right.$$

$$\left.-2\left(\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta\right)\cos\theta\right]$$

$$\frac{\partial g_{tt}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\Delta-a^2\sin^2\theta}{\Sigma}\right) = \frac{\partial}{\partial r}\left[\Sigma^{-1}\left(\Delta-a^2\sin^2\theta\right)\right]$$

$$= \Sigma^{-1}\frac{\partial\Delta}{\partial r}+\left(\Delta-a^2\sin^2\theta\right)(-1)\Sigma^{-2}\frac{\partial\Sigma}{\partial r}$$

$$= \frac{1}{\Sigma}\left(\frac{\partial\Delta}{\partial r}-\frac{\Delta-a^2\sin^2\theta}{\Sigma}\frac{\partial\Sigma}{\partial r}\right)$$

$$\frac{\partial g_{tt}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) = \frac{\partial}{\partial\theta}\left[\Sigma^{-1}\left(\Delta - a^2\sin^2\theta\right)\right]$$

$$= \Sigma^{-1}\left(\left(-a^2\right)(2)\sin\theta\cos\theta\right) + \left(\Delta - a^2\sin^2\theta\right)(-1)\Sigma^{-2}\frac{\partial\Sigma}{\partial\theta}$$

$$= \frac{1}{\Sigma}\left(\frac{a^2\sin^2\theta - \Delta}{\Sigma}\frac{\partial\Sigma}{\partial\theta} - 2a^2\sin\theta\cos\theta\right)$$

$$\frac{\partial g_{\phi t}}{\partial r} = \frac{\partial g_{t\phi}}{\partial r} = \frac{\partial}{\partial r}\left[\frac{1}{c^2}\frac{a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\Sigma}\right]$$

$$= \frac{1}{c^2}\frac{\partial}{\partial r}\left[\Sigma^{-1}\left(a\sin^2\theta\left(r^2 + a^2 - \Delta\right)\right)\right]$$

$$= \frac{1}{c^2}\left[\Sigma^{-1}\left(a\sin^2\theta\left(2r - \frac{\partial\Delta}{\partial r}\right)\right)\right.$$

$$\left. + a\sin^2\theta\left(r^2 + a^2 - \Delta\right)(-1)\Sigma^{-2}\frac{\partial\Sigma}{\partial r}\right]$$

$$= \frac{a\sin^2\theta}{c^2\Sigma}\left(2r - \frac{\partial\Delta}{\partial r} - \frac{r^2 + a^2 - \Delta}{\Sigma}\frac{\partial\Sigma}{\partial r}\right)$$

$$\frac{\partial g_{\phi t}}{\partial \theta} = \frac{\partial g_{t\phi}}{\partial \theta} = \frac{\partial}{\partial \theta}\left[\frac{1}{c^2}\frac{a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma}\right]$$

$$= \frac{1}{c^2}\frac{\partial}{\partial \theta}\left[\Sigma^{-1}\left(a\sin^2\theta\left(r^2+a^2-\Delta\right)\right)\right]$$

$$= \frac{1}{c^2}\Bigg[\Sigma^{-1}\left(r^2+a^2-\Delta\right)2a\sin\theta\cos\theta$$

$$+ a\sin^2\theta\left(r^2+a^2-\Delta\right)(-1)\Sigma^{-2}\frac{\partial \Sigma}{\partial \theta}\Bigg]$$

$$= \frac{a\sin\theta}{c^2\Sigma}\left[2\left(r^2+a^2-\Delta\right)\cos\theta - \frac{\sin\theta\left(r^2+a^2-\Delta\right)}{\Sigma}\frac{\partial \Sigma}{\partial \theta}\right]$$

These partial derivative equations can be programmed, and used to evaluate the elements of the Christoffel matrices needed to compute the polar coordinate Kerr relativistic accelerations in a computer simulation of test particle orbital motion around a Kerr black hole.

In the document located at http://www.sb635.qwestoffice.net/unified/matalg.pdf, I show how the above derivatives are used in the computation of the elements of the Christoffel matrices. There I also show how the inverse of the metric matrix is used in the geodesic equation computations. The Kerr metric matrix **G** given by Eq. (1) is block diagonal and symmetric. A 2×2 block symmetric matrix may be inverted using the following formula (see p. 29, Exercise 1.3.12, in *Introduction to Applied Mathematics* by G. Strang, Wellesley-Cambridge Press):

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}\mathbf{S}\mathbf{B}^T\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{S} \\ -\mathbf{S}\mathbf{B}^T\mathbf{A}^{-1} & \mathbf{S} \end{bmatrix}$$

where $\mathbf{S} = \left(\mathbf{C} - \mathbf{B}^T \mathbf{A}^{-1}\mathbf{B}\right)^{-1}$. For inverting **G** given by Eq. (1), we set

$$\mathbf{A} = \begin{bmatrix} g_{rr} & 0 \\ 0 & g_{\theta\theta} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} g_{\phi\phi} & g_{\phi t} \\ g_{t\phi} & g_{tt} \end{bmatrix}$$

and we see that

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{-1} \end{bmatrix}$$

where **0** is a 2×2 matrix of zeros. The matrix **A** is diagonal, so that $\mathbf{A}^{-1}$ is simply

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/g_{rr} & 0 \\ 0 & 1/g_{\theta\theta} \end{bmatrix}$$

The matrix **C** is just 2×2, so that its inverse is

$$\mathbf{C}^{-1} = \frac{1}{|\mathbf{C}|} \begin{bmatrix} g_{tt} & -g_{\phi t} \\ -g_{t\phi} & g_{\phi\phi} \end{bmatrix}$$

where the determinant $|\mathbf{C}| = g_{\phi\phi} g_{tt} - g_{\phi t} g_{t\phi} = g_{\phi\phi} g_{tt} - g_{\phi t}^2$. These equations can be programmed in order to compute the inverse of the Kerr metric matrix **G**, whose rows are needed for computing the elements of a Kerr Christoffel matrix.