A NUMERICAL SOLUTION OF THE RELATIVISTIC KEPLER PROBLEM

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One of the fundamental problems of orbital mechanics is the Kepler problem,[1] the calculation of the final position and velocity of an orbiting body (such as a planet orbiting a star) given its initial position and velocity and time of flight. In this column we discuss a solution to the relativistic Kepler problem using an extension of the numerical integration techniques commonly used for the nonrelativistic Kepler problem. We also discuss some interesting properties of relativistic orbits in a spherically symmetric gravitational field.

We first describe a particular solution to the nonrelativistic Kepler problem. A central mass is at the origin of an (x,y,z) rectangular coordinate system whose axes are fixed with respect to the distant stars. The planet moves around the central mass, which is assumed to be much larger than the mass of the planet. We say that the planet is a "test particle" that does not perturb the position of the central mass. For conical orbits, we choose a reference coordinate system known as a perifocal frame,[1] where the x axis passes through the origin and the perigee point (the point of closest approach) of the orbit (see Fig. 1). We assume that there is an observer who is stationary in the perifocal frame, and who can measure the positions and velocities of the moving test particle. Also, we assume that there is a clock attached to the test particle, and the observer can measure the time on this moving clock.

Given the test particle's initial state vector $[\mathbf{r}(0), \mathbf{v}(0)]$ and a time of flight t, we want to find the final state vector $[\mathbf{r}(t), \mathbf{v}(t)]$. If we assume that the central mass is a perfect sphere, the gravitational field surrounding the central mass is spherically symmetric, and the test particle's motion lies in the orbital plane. In this case the perifocal frame is convenient because all values of z are zero.

A common technique for solving the nonrelativistic Kepler problem is to integrate the acceleration vector $\mathbf{a}=d\mathbf{v}/dt=-(GM/r^3)\mathbf{r}$, where G is the gravitational constant, M is the central mass, and $r=|\mathbf{r}|$. Typically, a fourth-order Runge–Kutta integration scheme is used, although other numerical integration schemes would be sufficient. The total time of flight t is divided into small time steps to produce the desired accuracy. An advantage of using this approach (known as Cowell's method[1,2]) is that perturbations of the acceleration of a test particle can be introduced. For example, when our mission analysis group designs the low Earth orbits used as "park" orbits for geostationary communication satellite missions, our simulations take into account the true oblate spheroidal shape of the Earth.[3] This nonspherical shape sets up a nonspherical gravitational field. When simulating movement along the park orbit, the perturbed orbit is directly integrated using Cowell's method.

Although Cowell's method was conceived shortly after the turn of the century, its simplicity and numerical stability have led us to use it recently in an important targeting algorithm for an interplanetary mission to Mars.[4] We used Cowell's method of numerical integration to search along the park orbit for the point where the upper stage (which injected the spacecraft onto the required hyperbolic trajectory to Mars) should ignite. During the search, the program directly integrated the perturbed orbit due to the Earth's

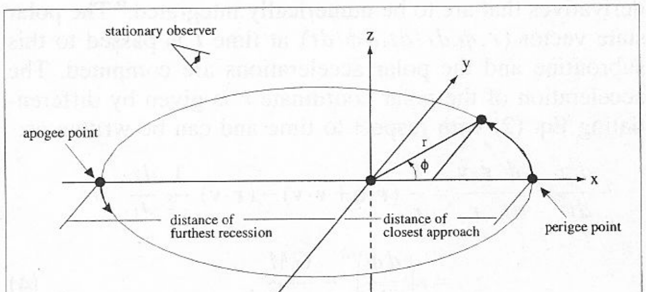

Figure 1. The perifocal coordinate frame. The apogee of the orbit is the point of furthest recession. The y axis passes through the origin, lies in the plane of the orbit, and is orthogonal to the x axis. The z axis passes through the origin and is orthogonal to the plane of the test particle's orbit.

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nonspherical shape. Even though Cowell's method is more computer intensive and not as numerically elegant as some other perturbation techniques, for example, Encke's method,[1,2] it still is popular. The continued use of Cowell's method is due in part to the availability of fast computers, both on the ground and in modern spacecraft.

After we consider the nonrelativistic problem, we will show that general relativistic effects produce perturbed accelerations, even around a perfectly spheroidal central mass and that Cowell's method can be used to compute the orbit.

We first transform the rectangular perifocal coordinates to perifocal polar coordinates (r,θ,ϕ). All motion occurs in the (x,y) plane, the plane of the orbit given by $\theta=\pi/2$. The latitudinal polar coordinate θ can be ignored, and the polar coordinate ϕ represents the in-plane longitudinal angular motion. The rectangular coordinates are related to the polar coordinates by

$$x=r\cos\phi,\quad y=r\sin\phi,\quad z=0, \tag{1}$$

where $r=\sqrt{x^2+y^2}$, and $\phi=\tan^{-1}(y/x)$ is the angle between the x axis and the position of the test particle (see Fig. 1).

Both the rectangular and polar coordinates are functions of time. The time derivative of r, the radial velocity, is given by

$$\frac{dr}{dt}=\frac{\mathbf{r}\cdot\mathbf{v}}{r}. \tag{2}$$

The time derivative of ϕ, the angular velocity, is given by

$$\frac{d\phi}{dt}=\frac{L_N}{r^2}, \tag{3}$$

where L_N is the angular momentum (per unit mass) of the orbit.[5] The angular momentum L_N is a constant of the motion and is given by $L_N=|\mathbf{r}\times\mathbf{v}|$. For spherically symmetric gravitational fields, L_N can be derived from any rectangular state of the orbit, for example, the initial state. Using Eqs. (1)–(3) and the initial rectangular state vector, we can compute the corresponding initial polar coordinates and their time derivatives.

Most programs utilize a subroutine for computing the derivatives that are to be numerically integrated.[6] The polar state vector $(r,\phi,dr/dt,d\phi/dt)$ at time t is passed to this subroutine and the polar accelerations are computed. The acceleration of the polar coordinate r is given by differentiating Eq. (2) with respect to time and can be written as:

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\frac{\mathbf{r}\cdot\mathbf{v}}{r}=\frac{1}{r}(\mathbf{r}\cdot\mathbf{a}+\mathbf{v}\cdot\mathbf{v})-(\mathbf{r}\cdot\mathbf{v})\frac{1}{r^2}\frac{dr}{dt}$$

$$=r\left(\frac{d\phi}{dt}\right)^2-\frac{GM}{r^2}, \tag{4}$$

where we have used $\mathbf{r}\cdot\mathbf{a}=-ra=-Gm/r$ and the relation $v^2=(dr/dt)^2+r^2(d\phi/dt)^2$.

The acceleration of the polar coordinate ϕ is given by differentiating Eq. (3):

$$\frac{d^2\phi}{dt^2}=\frac{d}{dt}\frac{|\mathbf{r}\times\mathbf{v}|}{r^2}=\frac{1}{r^2}\left(\frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|}\cdot(\mathbf{r}\times\mathbf{a}+\mathbf{v}\times\mathbf{v})\right)$$

$$-\frac{2}{r^3}|\mathbf{r}\times\mathbf{v}|\frac{dr}{dt}=-\frac{2}{r}\frac{|\mathbf{r}\times\mathbf{v}|}{r^2}\frac{dr}{dt}=-\frac{2}{r}\frac{d\phi}{dt}\frac{dr}{dt}. \tag{5}$$

Note that the polar accelerations d^2r/dt^2 and $d^2\phi/dt^2$ can be computed from the elements of the polar state given as $(r,\phi,dr/dt,d\phi/dt)$ and no information about the corresponding rectangular coordinates is needed for the solution of the Kepler problem.

Of course the use of polar coordinates is not necessary, and the only reason we have discussed the use of polar coordinates is that they are convenient for the relativistic Kepler problem. There are numerical techniques for the spherically symmetric, nonrelativistic Kepler problem that are much more efficient, for example, the method of universal variables.[1]

The relativistic Kepler problem

Given an initial state vector $[\mathbf{r}(0),\mathbf{v}(0)]$ and a time of flight t, the goal is to compute the final state vector $[\mathbf{r}(t),\mathbf{v}(t)]$ with all general relativistic effects included. For spherically symmetric gravitational fields, the motion of the test particle lies in the orbital plane, so that $\theta=\pi/2$. The difference from the nonrelativistic solution arises from the definition and derivation of the derivatives analogous to Eqs. (2)–(5).

In general relativity, time is incorporated as a mathematical dimension, so that the four-dimensional rectangular perifocal coordinates are $(x,y,0,t)$ and the four-dimensional polar coordinates are $(r,\pi/2,\phi,t)$. The coordinate time t is the time measured by the clock held stationary at the observer's location in the perifocal frame. The proper time τ is the time on the clock moving with the orbiting test particle as observed by the stationary observer in the perifocal frame. The proper time is smaller than the coordinate time due to time dilation. We say that the proper time has a "velocity" with respect to the coordinate time (and vice versa). These temporal velocities (time-dilation factors) play prominent roles in the derivation of the relativistic forms of the derivatives analogous to Eqs. (2)–(5).

The Schwarzschild metric,[7] written in terms of the proper-time differential, is used for defining the relativistic accelerations of test particles orbiting perfectly spheroidal and nonrotating central masses. The Schwarzschild metric is a special case of a more general metric known as the Kerr metric,[8–10] which is the correct metric to use for rotating central masses. The Schwarzschild metric in terms of the perifocal polar coordinates can be expressed as[7]

$$d\tau^2=-\frac{1}{c^2}\left(\frac{dr^2}{1-r_s/r}+r^2d\phi^2\right)+\left(1-\frac{r_s}{r}\right)dt^2. \tag{6}$$

As usual, c represents the speed of light, and $r_s=2GM/c^2$ is the Schwarzschild radius, the distance to the event horizon of a centrally located black hole.[7,10]

Equation (6), derived using standard differential geometry,[11] relates the square of an infinitesimal proper time interval to the squares of infinitesimal intervals in the radius, longitudinal angular motion, and the coordinate time. If we use Eq. (6) and more differential geometry, we can write the polar coordinate equations of motion as

$$\frac{d}{d\tau}\left(\frac{r}{r-r_s}\frac{dr}{d\tau}\right)+\frac{1}{2}\frac{r_s}{(r-r_s)^2}\left(\frac{dr}{d\tau}\right)^2-r\left(\frac{d\phi}{d\tau}\right)^2 +\frac{1}{2}\frac{r_s c^2}{r^2}\left(\frac{dt}{d\tau}\right)^2=0, \tag{7a}$$

$$\frac{d}{d\tau}\left(r^2\frac{d\phi}{d\tau}\right)=0, \tag{7b}$$

$$\frac{d}{d\tau}\left(\frac{r-r_s}{r}\frac{dt}{d\tau}\right)=0, \tag{7c}$$

From the equations of motion (7), we can derive the relativistic (proper-time) polar accelerations as follows. The relativistic radial acceleration is derived by solving for $d^2r/d\tau^2$ in Eq. (7a). The first term of Eq. (7a) can be written as

$$\frac{d}{d\tau}\left(\frac{r}{r-r_s}\frac{dr}{d\tau}\right)=\frac{r}{r-r_s}\frac{d^2r}{d\tau^2}+\frac{dr}{d\tau}\frac{d}{d\tau}\left(\frac{r}{r-r_s}\right) =\frac{r}{r-r_s}\frac{d^2r}{d\tau^2}-\frac{r_s}{(r-r_s)^2}\left(\frac{dr}{d\tau}\right)^2. \tag{8}$$

If we substitute Eq. (8) into Eq. (7a), we find

$$\frac{d^2r}{d\tau^2}=\frac{r-r_s}{r}\left[r\left(\frac{d\phi}{d\tau}\right)^2+\frac{1}{2}\frac{r_s}{(r-r_s)^2}\left(\frac{dr}{d\tau}\right)^2 -\frac{1}{2}\frac{r_s c^2}{r^2}\left(\frac{dt}{d\tau}\right)^2\right]. \tag{9}$$

The second derivative of the polar coordinate ϕ with respect to τ, that is, the relativistic angular acceleration, can be derived by solving for $d^2\phi/d\tau^2$ in Eq. (7b):

$$\frac{d^2\phi}{d\tau^2}=-\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}. \tag{10}$$

The second derivative of the coordinate time with respect to τ, that is, the relativistic temporal acceleration, is found by solving for $d^2t/d\tau^2$ in Eq. (7c). The result is

$$\frac{d^2t}{d\tau^2}=-\frac{r_s}{r(r-r_s)}\frac{dt}{d\tau}\frac{dr}{d\tau}. \tag{11}$$

We see from Eqs. (9), (10), and (11) that the relativistic polar-coordinate accelerations are expressed as functions of the first-order proper-time derivatives $dr/d\tau$, $d\phi/d\tau$, and $dt/d\tau$. Because

$$\frac{dr}{d\tau}=\frac{dr}{dt}\frac{dt}{d\tau} \quad\text{and}\quad \frac{d\phi}{d\tau}=\frac{d\phi}{dt}\frac{dt}{d\tau}, \tag{12}$$

we can compute these first-order relativistic derivatives using Eqs. (2) and (3) if we can compute $dt/d\tau$. The equation for $dt/d\tau$ can be obtained using the first integral of differential geometry:[7,10,11]

$$\frac{dt}{d\tau}=\left[\frac{r-r_s}{r}-\frac{r}{c^2(r-r_s)}\left(\frac{dr}{dt}\right)^2-\frac{r^2}{c^2}\left(\frac{d\phi}{dt}\right)^2\right]^{-1/2}. \tag{13}$$

Note that only derivatives with respect to the coordinate time appear in Eq. (13). The radial and angular velocities are assumed to be directly observable (or computable) and known to the stationary observer for any given coordinate-time polar state. Also note that Eqs. (9), (10), and (11) give the second-order proper-time derivatives of the coordinates r,ϕ, and t in terms of the elements of the relativistic polar state vector $(r,\phi,t,dr/d\tau,d\phi/d\tau,dt/d\tau)$. Hence we can use a standard integration routine to update the elements of the relativistic polar state vector.

The temporal velocity given by Eq. (13) is the general-relativistic extension of the special-relativistic time-dilation factor[7] $(1-v^2/c^2)^{-1/2}$. For special-relativistic conditions, a test particle obeys uniform and rectilinear motion (relative to the stationary observer), so that $(1-v^2/c^2)^{-1/2}$ is time invariant $(dv/dt=0)$. For the general-relastivistic case in which a test particle follows a curvilinear orbit (elliptical, parabolic, or hyperbolic) about a central mass, r fluctuates over the orbit, so that $dt/d\tau$ is not time invariant. Relative to the stationary observer, the passage of proper time speeds up, then slows down, etc., as the test particle orbits about the central mass and r changes in value. As discussed in the following, $dt/d\tau$ and $d\tau/dt$ play important roles in relating the nonrelativistic constants of motion to the relativistic constants of motion.

We can relate the nonrelativistic angular momentum L_N to the relativistic angular momentum L_R using $L_N=r^2(d\phi/dt)$ [see Eq. (3)] and $d\phi/dt=(d\phi/d\tau)(d\tau/dt)$ [see Eq. (12)]. We have

$$r^2\frac{d\phi}{dt}=r^2\frac{d\phi}{d\tau}\frac{d\tau}{dt} \quad\text{and hence}\quad L_N=L_R\frac{d\tau}{dt}, \tag{14}$$

where $L_R=r^2(d\phi/d\tau)$. As already noted, r generally fluctuates in value (except for circular orbits), so that the derivative $d\tau/dt$ also fluctuates. Equation (14) shows that the nonrelativistic angular momentum is not a constant of motion as determined by the stationary observer in the perifocal frame. Instead, it is the *relativistic* angular momentum L_R that is a constant of motion. If a quantity is constant with respect to the passage of proper time (such as L_R), and if $dt/d\tau$ is time varying (as it is for conical orbits whose eccentricities are nonzero), then this quantity cannot be constant with respect to the passage of coordinate time. This property provides a check on the correctness of the integration of the relativistic equations of motion. As a test particle is integrated around its orbit about a central mass, the integration must produce a nonrelativistic angular momentum that is periodic to the stationary observer, and the periodicity must satisfy Eq. (14).

If the nonrelativistic angular momentum L_N oscillates, the nonrelativistic specific energy[12] E_N given by

$$E_N=(e^2-1)\frac{(GM)^2}{2L_N^2}$$

also oscillates (e is the nonrelativistic eccentricity). Not only do L_N and E_N oscillate, so do all of the nonrelativistic "constants of motion," such as the semilatus rectum, the semimajor axis, the radii of perigee and apogee, and the eccentricity.

Numerical results

As an example, we discuss our results for a test particle orbiting a nonrotating, totally spheroidal ten-solar-mass black hole. In this case $M=10M_{\odot}$, where $M_{\odot}$ is the mass of the Sun. The results show some interesting characteristics of test particle movement in a Schwarzschild field.

The first step is to specify the $t=0$ value of the relativistic polar state vector $(r,\phi,t,dr/d\tau,d\phi/d\tau,dt/d\tau)$. If the initial position is given in rectangular coordinates, then it is straightforward to transform x and y to r and ϕ using $r=\sqrt{x^2+y^2}$ and $\phi=\tan^{-1}(y/x)$. (The inverse tangent must select the correct quadrant for ϕ as is done by the ATAN2 function in Fortran.) The last element of the initial polar vector $(dt/d\tau)$ is computed using Eq. (13), where the coordinate-time polar velocities are computed using Eqs. (2) and (3). Once the coordinate-time polar velocities and $dt/d\tau$ have been computed, the initial values of $dr/d\tau$ and $d\phi/d\tau$ are computed using Eq. (12).

We do the numerical integration in equal coordinate-time increments $\Delta t=t/n$, where t is the final time and n is the number of increments. For each Δt, the corresponding proper-time increment is

$$\Delta\tau=\Delta t\,\frac{d\tau}{dt}, \tag{15}$$

where $d\tau/dt$ is found by inverting the present value of $dt/d\tau$ at the beginning of each time step. The integration propagates the relativistic polar state vector forward by $\Delta\tau$. To the stationary observer, the full time step Δt passes on his/her stationary clock for each integration step, and an amount of time $\Delta\tau$ passes on the clock attached to the test particle.

A double-precision implementation of the fourth-order Runge–Kutta routine RK4 from the *Numerical Recipes*[6] set of Fortran programs was used. RK4 requires the use of a subroutine named DERIVS to compute the required derivatives. The subroutine statement of DERIVS was written as DERIVS(TIME,STATEVEC,DYDX) where the scalar argument TIME is the proper time, the array STATEVEC is the relativistic polar state $(r,\phi,t,dr/d\tau,d\phi/d\tau,dt/d\tau)$, and the array DYDX is the returned relativistic polar derivatives. Because the last three elements of STATEVEC are the first-order derivatives of (r,ϕ,t), DERIVS sets the first three elements of DYDX equal to the last three elements of STATEVEC. The second-order derivatives are computed using Eqs. (9), (10), and (11).

For the black hole of mass $M=10M_{\odot}$, the event horizon lies at the distance $r_s=29.533$ km. For the integration computations, it is convenient to measure all lengths in units of $r_s/2$, angles in radians, and time in seconds. A convenient choice for the initial state of the test particle is to start the simulation at either the perigee point or the apogee point of the first orbit. At these points the x velocity is zero, and only the y velocity is nonzero. In our calculation the initial state was chosen at the apogee point. The initial rectangular position vector was (40.0, 0, 0), and the initial coordinate-time velocity vector was (0, 2198.8785, 0). The orbit's initial Newtonian eccentricity was $e=0.530775$, where $e=|\mathbf{e}|$ and $\mathbf{e}=(1/GM)\,[(v^2-(GM/r)\mathbf{r}-(\mathbf{r}\cdot\mathbf{v})\mathbf{v}]$ with $v=|d\mathbf{r}/dt|$.

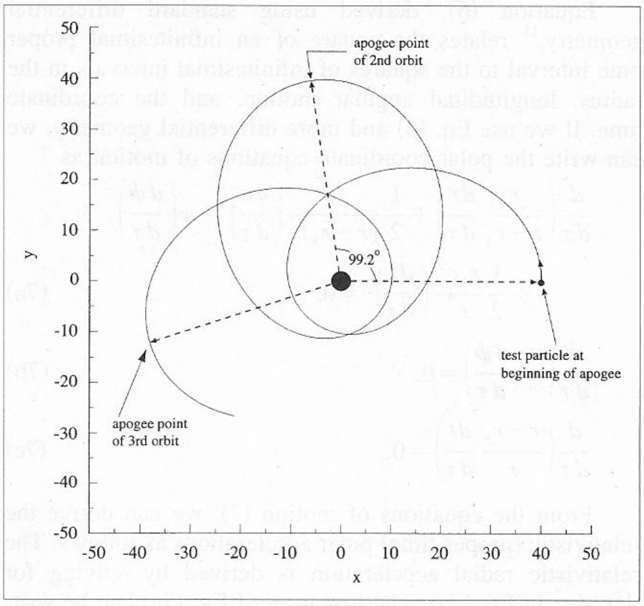

Figure 2. Plot of the test particle's orbital pathway in the (x,y) plane. Distances are measured in terms of $r_s/2$, where $r_s=29.533$ km.

From this initial state the following relativistic polar coordinate values and their proper time derivatives were computed: $r(0)=40$, $\phi(0)=0$, $t(0)=0$, $dr(0)/d\tau=0$, $d\phi(0)/d\tau=56.752$, $dt/d\tau=1.03237203$. The simulation was run for 0.195 s, terminating at slightly more than 5 1/4 orbits. The total coordinate time of flight was divided into 10 000 time steps. The view of the stationary observer for the first 3 1/4 orbits is given in Fig. 2 and is from above the orbital plane. From this point of view, the test particle started at the apogee of the first orbit and immediately began traveling towards the black hole. As the test particle moves, the Newtonian orbital elements oscillate as we discussed. The oscillation in the scalar Newtonian eccentricity e is plotted in Fig. 3 as a function of the coordinate time. Note how e oscillates as the test particle travels from apogee to perigee. The orbit is most circular at the perigee points, but reaches a maximum eccentricity at points immediately prior to and after the test particle reaches each apogee point. This regular oscillation in the eccentricity is due to the regular oscillations in the energy and angular momentum of the orbit. Due to the extreme relativistic effects, the direction of the apogee and perigee points shifts (due to precession) by 99.2°. A plot of the precession of the apogee and perigee points as a function of coordinate time is given in Fig. 4.

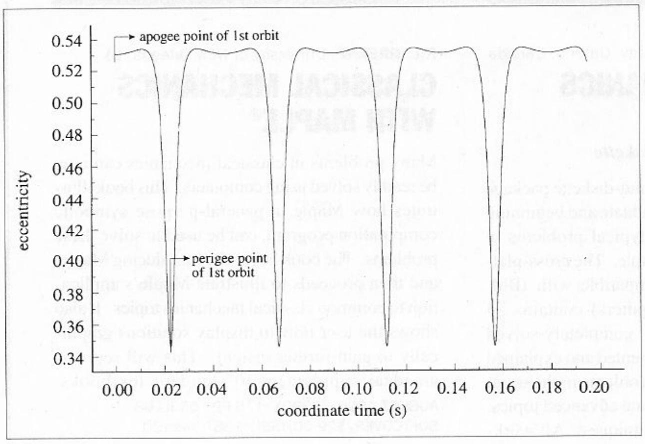

Figure 3. The orbital eccentricity versus the coordinate time (in seconds).

Given the solution to the relativistic Kepler problem, relativistic solutions to the second of the fundamental orbital-dynamics problems, the Lambert (or Gauss) problem,[1,4] can be formulated. In the Lambert problem, one attempts to find the initial velocity necessary to take an object between two given positions in a given amount of time. The relativistic Lambert problem solutions can be used as the foundation for solving relativistic targeting problems that will be required by future spacecraft traveling with velocities close to the speed of light and/or orbiting in intense gravitational fields such as those that exist near black holes.

Problems for further study

(1) Use classical kinematics to derive the equation $v^2=(dr/dt)^2+r^2(d\phi/dt)^2$.

(2) In the mathematics of the general theory, the relativistic velocities are given by $u^a=u^t v^a$, where u is understood as being a derivative with respect to τ, and v is a derivative with respect to t. As an example, if $a=r$, then $u^a=u^r=u^t v^r=(dt/d\tau)(dr/dt)$ [see Eq. (12)]. Also given is the equation of the first integral,[7,10,11] $1+g_{ab}u^a u^b=0$, where g_{ab} are the elements of the Schwarzschild metric tensor, and the Einsteinian summation rules hold for common indices. Given that $g_{rr}=r/c^2(r-r_s)$, $g_{\phi\phi}=r^2/c^2$, $g_{tt}=-(r-r_s)/r$ and all other off-diagonal metric tensor elements are zero, derive Eq. (13).

(3) An important test of general relativity is the precession of the perigee of Mercury's orbit. This precession can be quantified by finding the angle made between the initial major axis of the elliptical orbit and the final major axis of the orbit. Using the relativistic Kepler algorithm, confirm that Mercury's orbit precesses by 43″ during 100 years worth of motion.

(4) Use the relativistic Kepler algorithm to solve the following Lambert problem. For a nonrotating ten-solar-mass black hole, a test particle has the beginning rectangular position vector (40,0,0) in units of $r_s/2$. After 0.044 772 s of flight, it is at the position (−6.3837,39.4873,0). Find the initial and ending rectangular velocity vectors. [Answer: initial velocity vector is (0,2198.8785,0); final velocity vector is (−2170.8455, −349.9973,0).]

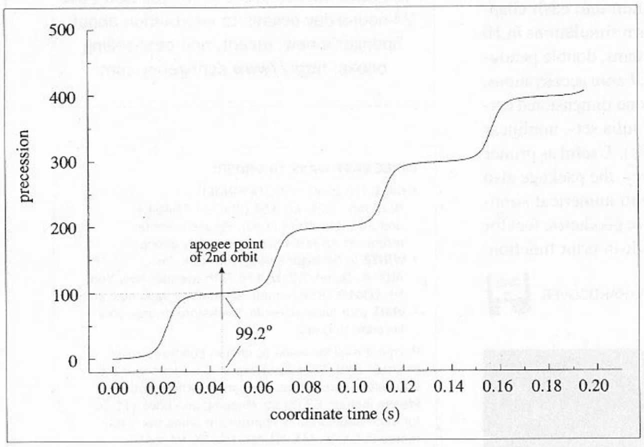

Figure 4. Perigee/apogee precession (in degrees) vs the coordinate time. Note that the rate of precession, given as tangents to the curve, is a maximum at the perigee points and is a minimum at the apogee points.

Acknowledgments

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