Simulating Relativistic Orbits About A Black Hole

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In a previous column,[1] we obtained numerical solutions of the relativistic motion of a particle orbiting a spherical, nonrotating central mass. In this supplement, we consider the more general case of a rotating central mass. This case is important because it is likely that the gravitational collapse that produces a black hole would yield one with nonzero rotational angular momentum.[2]

As before, we assume that the central mass (the black hole) is at the origin of a rectangular perifocal coordinate system.[1] We also assume that the axis of rotation of the central mass is oriented along the z-axis (see Fig. 1 of Ref. 1). Given a test particle's initial position and velocity and a time of flight t, our goal is to find its orbit. If the central mass does not rotate, the gravitational field is spherically symmetrical, the test particle's motion is in the orbital plane, and the perifocal frame is convenient because the value of z can be assumed to be zero. In contrast, if the central mass does rotate, its gravitational field is asymmetrical, and the test particle does not remain in the orbital plane. However, we shall see that the perifocal frame is still convenient.

As before, we assume that a stationary observer can measure the position and velocity of the test particle, and that there is a clock attached to the test particle. We first transform the rectangular coordinates of the test particle to polar coordinates using the relations $r = \sqrt{x^2 + y^2 + z^2}$, $\phi = \tan^{-1} y/x$, and $\theta = \cos^{-1} z/r$. The four-dimensional spacetime is defined by the square of the proper time differential $d\tau$:

$$(d\tau)^2 = (d\mathbf{x})^T \mathbf{G}\, d\mathbf{x}, \tag{1}$$

where $\mathbf{x} = (r, \theta, \phi, t)^T$ is the transpose of the column four vector. The proper time τ is the time on the clock attached to the test particle measured by the stationary observer, and the coordinate time t is the time on the stationary clock at the observer's location. The tensor $\mathbf{G}$ can be expressed as a 4×4 matrix:

$$\mathbf{G} = \begin{bmatrix} g_{rr} & g_{r\theta} & g_{r\phi} & g_{rt} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} & g_{\theta t} \\ g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} & g_{\phi t} \\ g_{tr} & g_{t\theta} & g_{t\phi} & g_{tt} \end{bmatrix}. \tag{2}$$

For non-Euclidean spacetimes,[2,3] the matrix elements of $\mathbf{G}$ are functions of $\mathbf{x}$, but for simplicity, we suppress this functional dependence. We also understand that each component of $\mathbf{x}$ is a function of τ.

In the general theory of relativity, the orbital motion of a test particle is given by the equations of a geodesic defined as the path of shortest distance in a four-dimensional spacetime.[2,3] The proper time equations of motion derived directly from the geodesic equations can be written in polar coordinates as

$$\frac{d^2r}{d\tau^2} = g^{rr}\left[\frac{1}{2}\frac{d\mathbf{x}^T}{d\tau}\frac{\partial\mathbf{G}}{\partial r}\frac{d\mathbf{x}}{d\tau} - \frac{dr}{d\tau}\frac{\partial g_{rr}}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{d\tau}\right] \tag{3}$$

$$\frac{d^2\theta}{d\tau^2} = g^{\theta\theta}\left[\frac{1}{2}\frac{d\mathbf{x}^T}{d\tau}\frac{\partial\mathbf{G}}{\partial\theta}\frac{d\mathbf{x}}{d\tau} - \frac{d\theta}{d\tau}\frac{\partial g_{\theta\theta}}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{d\tau}\right] \tag{4}$$

$$\frac{d^2\phi}{d\tau^2} = g^{\phi\phi}\left[\frac{1}{2}\frac{d\mathbf{x}^T}{d\tau}\frac{\partial\mathbf{G}}{\partial\phi}\frac{d\mathbf{x}}{d\tau} - \frac{d\phi}{d\tau}\frac{\partial g_{\phi\phi}}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{d\tau}\right] \tag{5}$$

$$\frac{d^2t}{d\tau^2} = g^{tt}\left[\frac{1}{2}\frac{d\mathbf{x}^T}{d\tau}\frac{\partial\mathbf{G}}{\partial t}\frac{d\mathbf{x}}{d\tau} - \frac{dt}{d\tau}\frac{\partial g_{tt}}{\partial\mathbf{x}}\cdot\frac{d\mathbf{x}}{d\tau}\right]. \tag{6}$$

The elements g^{rr}, $g^{\theta\theta}$, etc. in Eqs. (3)–(6) are the matrix elements of the inverse of $\mathbf{G}$. The proper time polar velocities in Eqs. (3)–(6) are given by $d\mathbf{x}/d\tau = (dt/d\tau)(d\mathbf{x}/dt)$, where the time dilation factor[1,3] $dt/d\tau$ is given by

$$\frac{dt}{d\tau} = \left(\frac{d\mathbf{x}^T}{dt}\mathbf{G}\frac{d\mathbf{x}}{dt}\right)^{-1/2}. \tag{7}$$

To solve Eqs. (3)–(7) we need to know the form of $\mathbf{G}$. Schwarzschild derived the elements of $\mathbf{G}$ for a nonrotating central mass. Kerr realized that the spacetime surrounding rotating black holes would be different, and derived the appropriate form of $\mathbf{G}$ for rotating black holes. His main result is that the mass of a rotating black hole is insufficient for describing its gravitational field, and that the angular momentum also plays a role. The matrix elements of $\mathbf{G}$ for a rotating black hole are functions of r and θ, its mass M, and its angular momentum J, and are given by:[2]

$$\begin{aligned} g_{rr} &= -\frac{\Sigma}{c^2\Delta}, \quad g_{\theta\theta} = -\frac{\Sigma}{c^2}, \quad g_{\phi\phi} = -\frac{1}{c^2}\left[\frac{\left(r^2+a^2\right)^2 - \Delta\, a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta \\ g_{tt} &= \frac{\Delta - a^2\sin^2\theta}{\Sigma}, \quad g_{t\phi} = g_{\phi t} = \frac{1}{c^2}\left[\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\right], \end{aligned} \tag{8}$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta \quad \text{and} \quad \Delta = r^2 - \frac{2GMr}{c^2} + a^2. \tag{9}$$

As usual, c is the speed of light, and G is Newton's gravitational constant. The characteristic length associated with the angular momentum is given by $a = J/(Mc)$. All other matrix elements of $\mathbf{G}$ besides those given in Eq. (8) are zero. Because the matrix elements in Eq. (8) are functions of r and θ only, the terms involving $\partial\mathbf{G}/\partial\phi$ and $\partial\mathbf{G}/\partial t$ in Eqs. (5) and (6) are zero, and the derivatives with respect to ϕ and t in $\partial g_{rr}/\partial\mathbf{x}$, $\partial g_{\theta\theta}/\partial\mathbf{x}$, $\partial g_{\phi\phi}/\partial\mathbf{x}$ and $\partial g_{tt}/\partial\mathbf{x}$ are zero. Note that if the central mass is not rotating, $J = 0$, and Eqs. (3)–(8) reduce to the Schwarzschild equations.[1,2,3]

Equations (2)–(8) provide the complete set of equations needed to simulate the Kerr orbits around rotating black holes. We use a double precision version of the fourth-order Runge-Kutta subroutine `RK4` given in *Numerical Recipes*.[4] This subroutine requires the subroutine `DERIVS` which computes the proper time accelerations given by Eqs. (3)–(6). We suggest that the nonzero partial derivatives in Eqs. (3)–(6) be calculated analytically using a symbolic manipulation program.

Suppose that a spacecraft (the test particle) is placed in orbit about a black hole of mass $M = 10M_\odot$, where $M_\odot$ is the mass of the Sun. Another convenient characteristic length, called the *Schwarzschild radius*, is given by[1] $r_s = 2GM/c^2$. (For $M = 10M_\odot$, $r_s = 29.533\,\text{km}$.) From Eq. (8), we see that g_{rr} is undefined for $\Delta = 0$. The value of r at which $\Delta = 0$ is called the *Kerr radius* r_k and is given by $r_k = m + \sqrt{m^2 - a^2}$, where $m = r_s/2$. Because r_k must be real and positive, we see that $a \leq m$. Because a must be proportional to m, we set $a = pm$, where $0 \leq p \leq 1$. In the following, we have set $p = 0.5$, half-way between its minimum and maximum values, which implies a fairly rapidly rotating black hole. In the following, we measure lengths in units of m, angles in radians, and time in seconds.

We consider a spacecraft with an apogee point given by $\mathbf{x} = (25.0, \pi/2, 0.0, 0.0)^T$. The spacecraft at this point has a velocity such that the instantaneous plane of its motion makes a 45° angle with the xy-plane, that is, the inclination of the orbit at this point is 45°. The proper time velocity at this point is $d\mathbf{x}^T/d\tau = (dr/d\tau, d\theta/d\tau, d\phi/d\tau, dt/d\tau)^T$, with $dr/d\tau = 0$, $d\theta/d\tau = -85.58610183$, $d\phi/d\tau = 85.58610183$, and $dt/d\tau = 1.0539077376$. These values for $\mathbf{x}$ and $d\mathbf{x}^T/d\tau$ correspond to an initial speed of approximately 0.14 c and a Newtonian eccentricity of $e = 0.5$. At this speed, the spacecraft completes one revolution about the black hole in approximately 0.013 s of proper time. To see the relativistic effects on the spacecraft's orbit, we consider the spacecraft's motion for 0.04 s of proper time. The proper time step should be no greater than 10^{-5} s.

The view of the stationary observer for the first 0.04 s (proper time) of the spacecraft's motion is shown in Fig. 1. The total amount of coordinate time that passed on the stationary clock was 0.04593 s. The difference is due to time dilation. Note the large precession of the major axes of the first and second orbits caused by the extreme relativistic effects. Due to its rotation, the black hole's gravitational field is asymmetrical because $g_{t\phi}$ and $g_{\phi t}$ are nonzero everywhere. This asymmetry causes the spacecraft to continually come out of the plane producing inclination shifts of its orbits. These inclination shifts are better seen from the perspective of the stationary observer as shown in Fig. 2, where the spacecraft has been propagated for an additional time of 0.042 s. If the spacecraft's motion is considered for an extended time, its trajectory would form a torus around the rotating black hole. The spacecraft could then explore a volume of space surrounding the black hole without initiating any orbit-changing maneuvers. In contrast, Newtonian mechanics implies that the spacecraft would need to perform rocket engine burns to change its inclination.

Suggestions for further study

(1) Derive closed form solutions for the partial derivatives needed in Eqs. (3)–(6).

(2) Suppose that a test particle's initial position is $(25.0, 0.0, 0.0)$ in rectangular coordinates. Its velocity is orthogonal to this position vector and makes a 45° angle with the xy-plane. The test particle's coordinate time speed is 0.1 c. The central mass is a black hole with a mass of $M = 10M_{\odot}$, and the dimensionless angular momentum of $a = 0.5\,m$. Predict the orbit of the test particle.

(3) Suppose that the spacecraft has completed its scientific mission, and needs to escape from the black hole to travel to its next destination. According to classical Newtonian mechanics, the procedure would be for the spacecraft to do an engine burn at the appropriate point on its last bound orbit (usually the perigee point). After the burn, the spacecraft would be injected to a hyperbolic orbit whose asymptote points in the desired escape direction.[5] The escape orbit's eccentricity should be greater than unity, so that the spacecraft would no longer be bound to the black hole. Show that, due to relativistic effects, the burn would actually not have to immediately produce an eccentricity greater than

unity for the spacecraft to eventually escape. Produce a three-dimensional plot showing the pathway of the spacecraft before and after the escape-producing engine burn. (Assume that the engine burn produces an instantaneous increase in the velocity of the spacecraft at the perigee point of one the orbits depicted in Fig. 1.)

(4) Some researchers[2,3,6] have stated that no stable circular orbits exist around Schwarzschild (nonrotating) black holes for $3m \leq r_c \leq 6m$, where r_c is the radius of the circular orbit. That is, a stationary observer cannot see a test particle continually circle a Schwarzschild black hole at a radius in this interval. Use the Kerr equations with $a = 0$ and show that this implication is incorrect. Hint: choose an initial state vector whose Newtonian eccentricity is zero. Explain why all continually circular orbits that lie above the event horizon are stable.

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References

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Figure Captions

Fig. 1. Three-dimensional plot of the spacecraft's orbital path. Distances are measured in units of $m = 14.767$ km. The precession at the beginning of the second orbit is $276.7°$.

Fig. 2. View of the spacecraft's path showing the shifts of the inclination of its orbit due to the rotation of the black hole.